

# SEMISTABLE HIGGS BUNDLES AND REPRESENTATIONS OF ALGEBRAIC FUNDAMENTAL GROUPS: POSITIVE CHARACTERISTIC CASE

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**ABSTRACT.** Let  $k$  be an algebraic closure of finite fields with odd characteristic  $p$  and a smooth projective scheme  $\mathbf{X}/W(k)$ . Let  $\mathbf{X}^0$  be its generic fiber and  $X$  the closed fiber. For  $\mathbf{X}^0$  a curve Faltings conjectured that semistable Higgs bundles of slope zero over  $\mathbf{X}_{\mathbb{C}_p}^0$  correspond to genuine representations of the algebraic fundamental group of  $\mathbf{X}_{\mathbb{C}_p}^0$  in his  $p$ -adic Simpson correspondence [3]. This paper intends to study the conjecture in the characteristic  $p$  setting. Among other results, we show that isomorphism classes of rank two semistable Higgs bundles with trivial chern classes over  $X$  are associated to isomorphism classes of two dimensional genuine representations of  $\pi_1(\mathbf{X}^0)$  and the image of the association contains all irreducible crystalline representations. We introduce intermediate notions *strongly semistable Higgs bundles* and *quasi-periodic Higgs bundles* between semistable Higgs bundles and representations of algebraic fundamental groups. We show that quasi-periodic Higgs bundles give rise to genuine representations and strongly Higgs semistable are equivalent to quasi-periodic. We conjecture that a Higgs semistable bundle is indeed strongly Higgs semistable.

## 1. INTRODUCTION

N. Hitchin [4] introduced rank two stable Higgs bundles over a compact Riemann surface  $X$  and showed that they correspond naturally to irreducible representations of the fundamental group  $\pi_1(X)$  by solving a Yang-Mills equation, which generalizes the earlier works by Donaldson, Uhlenbeck-Yau for polystable vector bundles. Later C. Simpson obtained the full correspondence for any polystable Higgs bundles over arbitrary dimensional complex projective manifolds. In [3] G. Faltings established the correspondence between Higgs bundles and generalized representations of  $\pi_1(X)$  over  $p$ -adic fields. He conjectured that semistable Higgs bundles under his functor shall correspond to usual  $p$ -adic representations of  $\pi_1(X)$ . In this paper we intend to study Faltings's conjecture in the characteristic  $p$  setting.

Let  $k$  be the algebraic closure of finite fields of odd characteristic  $p$ . Let  $\mathbf{X}/W(k)$  be a smooth projective  $W := W(k)$ -scheme and  $X/k$  its closed fiber. In this paper, if not specified, a Higgs bundle over  $X$  means a system of Hodge bundles

$$(E = \oplus_{i+j=n} E^{i,j}, \theta = \oplus_{i+j=n} \theta^{i,j}),$$

where  $E$  is a vector bundle over  $X$ ,  $\theta$  is a morphism of  $\mathcal{O}_X$ -modules satisfying

$$\theta^{i,j} : E^{i,j} \rightarrow E^{i-1,j+1} \otimes \Omega_X, \quad \theta \wedge \theta = 0.$$

For simplicity, we assume throughout that  $n \leq p-2$ . Fix an ample divisor  $\mathbf{H} \subset \mathbf{X}$  over  $W$ . The Higgs semistability of  $(E, \theta)$  is referred to the  $\mu$ -semistability with respect to  $H \subset X$ , the reduction of  $\mathbf{H}$ .

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This work is supported by the SFB/TR 45 'Periods, Moduli Spaces and Arithmetic of Algebraic Varieties' of the DFG, and partially supported by the University of Science and Technology of China.

**Theorem 1.1** (Corollary 3.9 and Corollary 4.2). *There is a functor from the category of quasi-periodic Higgs-de Rham sequences of type  $(e, f)$  to the category of crystalline representations of  $\pi_1(\mathbf{X}'^0)$  into  $\mathrm{GL}(\mathbb{F}_{p^f})$ , where  $\mathbf{X}'^0$  is the generic fiber of  $\mathbf{X}' := \mathbf{X} \times_W \mathcal{O}_K$  for a totally ramified extension  $\mathrm{Frac}(W) \subset K$  with ramification index  $e$ . There is also a functor in the opposite direction. These two functors are equivalence of categories in the case  $e = 0$  and quasi-inverse to each other.*

Consequently, we obtain the following

**Corollary 1.2** (Corollary 5.2). *Under the above functors, there is one to one correspondence between the isomorphism classes of irreducible crystalline  $\mathbb{F}_{p^f}$ -representations of  $\pi_1(\mathbf{X}^0)$  and the isomorphism classes of periodic Higgs stable bundles of period  $f$ .*

The leading term of a quasi-periodic Higgs-de Rham sequence is a quasi-periodic Higgs bundle. We show that

**Theorem 1.3** (Theorem 2.5). *A quasi-periodic Higgs bundle is strongly Higgs semistable with trivial chern classes. Conversely, A strongly Higgs semistable bundle with trivial chern classes is quasi-periodic.*

Strongly semistable vector bundles are strongly semistable Higgs bundles with trivial Higgs fields. As a semistable bundle need not be strongly semistable, the notion of strongly semistability should be replaced by the strongly Higgs semistability. The next result supports our viewpoint.

**Theorem 1.4** (Theorem 2.6). *A rank two semistable Higgs bundle is strongly Higgs semistable.*

We would like to make the following

**Conjecture 1.5.** *A semistable Higgs bundle is strongly Higgs semistable.*

As an application of the above results, we obtain the following

**Corollary 1.6** (Theorem 5.6). *Any isomorphism class of rank two semistable Higgs bundles with trivial chern classes over  $X$  is associated to an isomorphism class of crystalline representations of  $\pi_1(\mathbf{X}^0)$  into  $\mathrm{GL}_2(k)$ . The image of the association contains all irreducible crystalline representations of  $\pi_1(\mathbf{X}^0)$  into  $\mathrm{GL}_2(k)$ .*

The plan of our paper is arranged as follows: in Section 2 we introduce the notions *strongly Higgs semistable bundles* which generalizes the notion of strongly semistable vector bundles in the paper [7] of Lange-Stuhler and *quasi-periodic Higgs bundles* which generalizes the notion of periodic Higgs subbundles introduced in [11]. We show that a strongly Higgs semistable with trivial chern classes is equivalent to a quasi-periodic Higgs bundle, and a rank two semistable Higgs bundle is strongly Higgs semistable. We conjecture that semistable Higgs bundles of arbitrary rank are strongly Higgs semistable. In Section 3 we show in Theorem 3.1 that there is a one to one correspondence between the strict  $p$ -torsion category  $\mathcal{MF}_{[0,n],f}^\nabla(\mathbf{X}/W)$  of Faltings with endomorphism  $\mathbb{F}_{p^f}$  and the category of periodic Higgs-de Rham sequences of type  $(0, f)$ . In Section 4, we extend the construction for periodic Higgs bundles to quasi-periodic Higgs bundles. In Section 5, we give some complements and applications of the above theory.

**Acknowledgements:** Arthur Ogus has recently pointed to us that the inverse Cartier transform in the paper [13] for the nilpotent Higgs bundles coincides with the construction in [9]. Christopher Deninger has drawn our attention to the work [6], and Adrian Langer has helped us understanding [6]. We thank them heartily.

## 2. STRONGLY SEMISTABLE HIGGS BUNDLES

In this paper, a vector bundle over  $X$  means a torsion free coherent sheaf of  $\mathcal{O}_X$ -module. A Higgs-de Rham sequence over  $X$  is a sequence of form

$$\begin{array}{ccccc} & (H_0, \nabla_0) & & (H_1, \nabla_1) & \\ C_0^{-1} \nearrow & & Gr_{Fil_0} \searrow & C_0^{-1} \nearrow & Gr_{Fil_1} \searrow \\ (E_0, \theta_0) & & (E_1, \theta_1) & & \dots \end{array}$$

In the sequence,  $C_0^{-1}$  is the inverse Cartier transform constructed in [13] (see also [9]). A. Ogus remarked that the exponential twisting of [9] is equivalent to the more general construction in [13] and the equivalence is implicitly implied by Remark 2.10 loc. cit..  $Fil_i$  is a decreasing filtration on  $H_i$  with the property  $Fil_i^0 = H_i$  and  $Fil_i^{n+1} = 0$  and such that  $\nabla_i$  obeys the Griffiths transversality with respect to it.

**Definition 2.1.** A Higgs bundle  $(E, \theta)$  is called strongly Higgs semistable if it appears in the leading term of a Higgs-de Rham sequence whose Higgs terms  $(E_i, \theta_i)$ s are all Higgs semistable.

Recall that [7] a vector bundle  $E$  is said to be strongly semistable if  $F_X^{*n}E$  is semistable for all  $n \in \mathbb{N}$ . Clearly, a strongly semistable vector bundle  $E$  is strongly Higgs semistable: one takes simply the Higgs-de Rham sequence as

$$\begin{array}{ccccc} & (F_X^*E, \nabla_{can}) & & (F_X^{*2}E, \nabla_{can}) & \\ C_0^{-1} \nearrow & & Gr_{Fil_{tr}} \searrow & C_0^{-1} \nearrow & Gr_{Fil_{tr}} \searrow \\ (E_0, 0) & & (F_X^*E, 0) & & \dots \end{array}$$

where  $\nabla_{can}$  is the canonical connection in the theorem of Cartier descent and  $Fil_{tr}$  is the trivial filtration.

**Definition 2.2.** A Higgs bundle  $(E, \theta)$  is called periodic if it appears in the leading term of a periodic Higgs-de Rham sequence, that is, there exists a natural number  $f$  such that there is an isomorphism of Higgs bundles

$$(E_f, \theta_f) \cong (E_0, \theta_0),$$

which via  $C_0^{-1}$  induces inductively a filtered isomorphism of de Rham bundles

$$(H_{f+i}, \nabla_{f+i}, Fil_{f+i}) \cong (H_i, \nabla_i, Fil_i),$$

and hence also an isomorphism of Higgs bundles for all  $i \in \mathbb{N}$ ,

$$(E_{f+i}, \theta_{f+i}) \cong (E_i, \theta_i).$$

The minimal number  $f \geq 1$  is called the period of the sequence. One understands a periodic Higgs-de Rham sequence of period  $f$  through the following diagram:

$$\begin{array}{ccccc} & (H_0, \nabla_0) & & (H_{f-1}, \nabla_{f-1}) & \\ C_0^{-1} \nearrow & & Gr_{Fil_0} \searrow & C_0^{-1} \nearrow & Gr_{Fil_{f-1}} \searrow \\ (E_0, \theta_0) & & \dots & & (E_f, \theta_f) \\ & & \cong & & \end{array}$$

In general, we make the following

**Definition 2.3.** A Higgs bundle  $(E, \theta)$  is called quasi-periodic if it appears in the leading term of a quasi-periodic Higgs-de Rham sequence, i.e., it becomes periodic after a nonnegative integer  $e \geq 0$ .

We add a simple lemma which follows directly from the construction of  $C_0^{-1}$  via the exponential function [9].

**Lemma 2.4.** *Let  $(E, \theta)$  be a nilpotent Higgs bundle (not necessary a system of Hodge bundles) with exponent  $\leq p-1$ . It holds that  $\det C_0^{-1}(E, \theta) = F_X^* \det E$ . Consequently,*

$$\deg C_0^{-1}(E, \theta) = p \deg E.$$

*Proof.* It follows from the fact that in the determinant, the exponential twisting appeared in the construction of  $C_0^{-1}(E, \theta)$  is simply the identity.  $\square$

**Theorem 2.5.** *A quasi-periodic Higgs bundle is strongly Higgs semistable with trivial chern classes. Conversely, a strongly Higgs semistable bundle with trivial chern classes is quasi-periodic.*

*Proof.* One observes that, in a Higgs-de Rham sequence,  $c_l(E_{i+1}) = p^l c_l(E_i), i \geq 0$ . This forces the chern classes of a quasi-periodic Higgs bundle to be trivial. By Lemma 2.4, a degree  $\lambda$  Higgs subbundle (not necessarily subsystem of Hodge bundles) in  $(E_i, \theta_i)$  gives rise to a degree  $p\lambda$  Higgs subbundle in  $(E_{i+1}, \theta_{i+1})$ . This implies that, in a Higgs-de Rham sequence of a quasi-periodic Higgs bundle, each Higgs term  $(E_i, \theta_i)$  contains no Higgs subbundle of positive degree. So  $(E_i, \theta_i)$  is Higgs semistable. Thus we have shown the first statement.

Assume  $X$  has a model over a finite field  $k' \subset k$ . Let  $M_{r,ss}(X)$  be the moduli space of  $S$ -equivalence classes of rank  $r$  semistable Higgs bundles with trivial chern classes over  $X$ . After A. Langer [6] and C. Simpson [8], it is a projective variety over  $k'$ . For a strongly Higgs semistable bundle  $(E, \theta)$  over  $X$  with trivial chern classes, we consider the set of  $S$ -isomorphism classes  $\{[(E_i, \theta_i)], i \in \mathbb{N}_0\}$ , where  $(E_i, \theta_i)$ s are all Higgs terms in a Higgs-de Rham sequence for  $(E, \theta)$ . Note that the operators  $C_0^{-1}$  and  $Gr_{Fil_i}$  do not change the definition field of objects. Thus, if the leading term  $(E_0, \theta_0) = (E, \theta)$  is defined over a finite field  $k'' \supset k'$ , all terms in a Higgs-de Rham sequence are defined over  $k''$ . This implies that the above sequence is a sequence of  $k''$ -rational points in  $M_{r,ss}(X)$  and hence finite. So we find two integers  $e$  and  $f$  such that  $[(E_e, \theta_e)] = [(E_{e+f}, \theta_{e+f})]$ . If  $(E_e, \theta_e)$  is Higgs stable, then there is a  $k''$ -isomorphism of Higgs bundles  $(E_e, \theta_e) \cong (E_{e+f}, \theta_{e+f})$ . If it is only Higgs semistable, we obtain only a  $k''$ -isomorphism between their gradings. But we do find a  $k'''$ -isomorphism of Higgs bundles after a certain finite field extension  $k'' \subset k'''$ : there exists a finite field extension  $k'''$  of  $k''$  such that  $(E_e, \theta_e)$  admits a Jordan-Hölder (abbreviated as JH) filtration defined over  $k'''$ . The operator  $Gr_{Fil_e} \circ C_0^{-1}$  transports this JH filtration into a JH filtration on  $(E_{e+1}, \theta_{e+1})$  defined over the same field  $k'''$ . Then this holds for any Higgs term  $(E_i, \theta_i), i \geq e$ . Without loss of generality, we assume that there are only two stable components in the gradings. Then the isomorphism classes of extensions over two stable Higgs bundles are described by a projective space over a finite field. Since there are finitely many  $S$ -equivalence classes in  $\{(E_i, \theta_i), i \geq e\}$  and over each  $S$ -equivalence class there are only finite many  $k'''$ -isomorphism classes, there exists a  $k'''$ -isomorphism  $(E_e, \theta_e) \cong (E_{e+f}, \theta_{e+f})$  after possibly choosing another  $e, f$ . It determines via  $C_0^{-1}$  an isomorphism of flat bundles between  $(H_e, \nabla_e)$  and  $(H_{e+f}, \nabla_{e+f})$ . This isomorphism defines a filtration  $Fil'_{e+f}$  on  $H_{e+f}$  from the filtration  $Fil_e$  on  $H_e$ , which may differs

from the original one. Put

$$(E'_{e+f+1}, \theta'_{e+f+1}) = Gr_{Fil'_{e+f}}(H_{e+f}, \nabla_{e+f}).$$

One has then a tautological isomorphism between  $(E_{e+1}, \theta_{e+1})$  and  $(E'_{e+f+1}, \theta'_{e+f+1})$ . Continuing the construction, we show that a strongly semistable Higgs bundle with trivial chern classes can be putted into the leading term of a quasi-periodic Higgs-de Rham sequence, hence quasi-periodic. This shows the converse statement.  $\square$

**Theorem 2.6.** *A rank two semistable Higgs bundle is strongly Higgs semistable.*

*Proof.* Let  $(E, \theta)$  be a rank two semistable Higgs bundle over  $X/k$ . Note first that, for the reason of rank,  $\theta^2 = 0$ . Hence the operator  $C_0^{-1}$  applies. Denote  $(H, \nabla)$  for  $C_0^{-1}(E, \theta)$ , and  $HN$  the Harder-Narasimhan filtration on  $H$ . We need to show that the graded Higgs bundle  $Gr_{HN}(H, \nabla)$  is semistable. If  $H$  is semistable, there is nothing to prove: in this case, the  $HN$  is trivial and hence the induced Higgs field is zero, and  $Gr_{HN}(H, \nabla) = (H, 0)$  is Higgs semistable. Otherwise, the  $HN$  filtration is of form

$$0 \rightarrow L_1 \rightarrow H \rightarrow L_2 \rightarrow 0.$$

**Claim 2.7.**  $L_1 \subset H$  is not  $\nabla$ -invariant.

*Proof.* We can assume that  $\theta \neq 0$ . Otherwise, by the Cartier descent, it follows that  $L_1 \cong F_X^* G_1$  for a rank one sheaf  $G_1 \subset E$  whose degree is positive, which contradicts with the semistability of  $E$ . Write  $E = E^{1,0} \oplus E^{0,1}$  and  $\theta : E^{1,0} \rightarrow E^{0,1} \otimes \Omega_X$  is nonzero. By the local construction of  $C_0^{-1}$ , the  $p$ -curvature of  $\nabla$  is nilpotent and nonzero. As  $L_1$  is of rank one, it follows that the  $p$ -curvature of  $\nabla|_{L_1}$  is zero. Again by the construction of  $C_0^{-1}$ ,  $\nabla$  preserves the rank one subsheaf  $L'_1 := C_0^{-1}(E^{0,1}, 0)$  and the restriction  $\nabla|_{L'_1}$  has also the  $p$ -curvature zero property. Let  $C \subset X$  be a generic curve. Then the nonzeroness of  $\theta$  implies that  $E^{0,1}|_C$  has negative degree. So is  $L'_1|_C$ . As  $L_1$  has positive degree, they are not the same rank one subsheaf of  $H$ . Therefore, over a nonempty open subset  $U \subset C$ , one has  $H = L_1 \oplus L'_1$ . It contradicts the nonzeroness of the  $p$ -curvature of  $\nabla$ .  $\square$

Then it follows that

$$\theta' = Gr_{HN} \nabla : L_1 \rightarrow L_2 \otimes \Omega_X$$

is nonzero. Let  $L \subset Gr_{HN} H = L_1 \oplus L_2$  be a Higgs sub line bundle. As  $\theta'|_L = 0$ , the composite

$$L \hookrightarrow L_1 \oplus L_2 \twoheadrightarrow L_1$$

is zero. Hence the natural map  $L \rightarrow L_2$  is nonzero and it follows that

$$\deg L \leq \deg L_2 < 0.$$

In this case,  $Gr_{HN}(H, \nabla)$  is Higgs stable.  $\square$

We would like to make the following

**Conjecture 2.8.** *A semistable Higgs bundle is strongly Higgs semistable.*

## 3. A HIGGS CORRESPONDENCE

In this section we aim to establish a Higgs correspondence between the category of Higgs-de Rham sequences of periodic Higgs bundles over  $X/k$  and the (modified) strict  $p$ -torsion category  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$ ,  $n \leq p-2$  (abbreviated as  $\mathcal{MF}$ ) introduced by Faltings [1]. Here strict means that each object in the category is annihilated by  $p$ .

We introduce first the category  $\mathcal{MF}_{[0,n],f}^\nabla(\mathbf{X}/W)$ , a modification of the Faltings category  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$ . For each  $f \in \mathbb{N}$ , let  $\mathbb{F}_{p^f}$  be the unique extension of  $\mathbb{F}_p$  in  $k$  of degree  $f$ . An object in  $\mathcal{MF}_{[0,n],f}^\nabla(\mathbf{X}/W)$  (abbreviated as  $\mathcal{MF}_f$ ) is a five tuple  $(H, \nabla, Fil, \Phi, \iota)$ , where  $(H, \nabla, Fil, \Phi)$  is object in  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$  and

$$\iota : \mathbb{F}_{p^f} \hookrightarrow \text{End}_{\mathcal{MF}}(H, \nabla, Fil, \Phi)$$

is an embedding of  $\mathbb{F}_p$ -algebras. A morphism is a morphism in  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$  respecting the endomorphism structure. Clearly, the category  $\mathcal{MF}_{[0,n],f}^\nabla(\mathbf{X}/W)$  for  $f = 1$  is just the original  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$ . On the Higgs side, we define the category  $\mathcal{HB}_{n,(0,f)}(X/k)$  (abbreviated as  $\mathcal{HB}_{(0,f)}$ ) of the periodic Higgs-de Rham sequences of type  $(0, f)$  as follows: an object is a tuple  $(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi)$  where  $(E, \theta)$  is a Higgs bundle on  $X/k$ ,  $Fil_i$ ,  $0 \leq i \leq f-1$  is a decreasing filtration on  $C_0^{-1}(E_i, \theta_i)$  satisfying  $Fil_i^0 = C_0^{-1}(E_i, \theta_i)$ ,  $Fil_i^{n+1} = 0$  and the Griffiths transversality such that  $Gr_{Fil_i}(H_i, \nabla_i)$  is torsion free with  $(E_0, \theta_0) = (E, \theta)$  and  $(E_i, \theta_i) := Gr_{Fil_{i-1}}(H_{i-1}, \nabla_{i-1})$  inductively defined, and  $\phi$  is an isomorphism of Higgs bundles

$$Gr_{Fil_{f-1}} \circ C_0^{-1}(E_{f-1}, \theta_{f-1}) \cong (E, \theta).$$

The information of such a tuple is encoded in the following diagram:

$$\begin{array}{ccccc}
 & (H_0, \nabla_0) & & (H_{f-1}, \nabla_{f-1}) & \\
 & \nearrow C_0^{-1} & \searrow Gr_{Fil_0} & \nearrow C_0^{-1} & \searrow Gr_{Fil_{f-1}} \\
 (E_0, \theta_0) & & \dots & & (E_f, \theta_f) \\
 & \searrow \phi & & \nearrow \phi & \\
 & \cong & & \cong & 
 \end{array}$$

Note that  $(E, \theta)$  of a tuple in the category is indeed periodic. A morphism between two objects is a morphism of Higgs bundles respecting the additional structures. As an illustration, we explain a morphism in the category  $\mathcal{HB}_{(0,1)}$  in detail: let  $(E_i, \theta_i, Fil_i, \phi_i)$ ,  $i = 1, 2$  be two objects and

$$f : (E_1, \theta_1, Fil_1, \phi_1) \rightarrow (E_2, \theta_2, Fil_2, \phi_2)$$

a morphism. By the functoriality of  $C_0^{-1}$ , the morphism  $f$  of Higgs bundles induces a morphism of flat bundles:

$$C_0^{-1}(f) : C_0^{-1}(E_1, \theta_1) \rightarrow C_0^{-1}(E_2, \theta_2).$$

It is required to be compatible with the filtrations, and the induced morphism of Higgs bundles is required to be compatible with  $\phi$ s, that is, there is a commutative diagram

$$\begin{array}{ccc}
 Gr_{Fil_1} C_0^{-1}(E_1, \theta_1) & \xrightarrow{\phi_1} & (E_1, \theta_1) \\
 Gr C_0^{-1}(f) \downarrow & & \downarrow f \\
 Gr_{Fil_2} C_0^{-1}(E_2, \theta_2) & \xrightarrow{\phi_2} & (E_2, \theta_2).
 \end{array}$$

**Theorem 3.1.** *There is a one to one correspondence between the category  $\mathcal{MF}_{[0,n],f}^\nabla(\mathbf{X}/W)$  and the category  $\mathcal{HB}_{n,(0,f)}(X/k)$ .*

To show the theorem, we choose and fix a small affine covering  $\{\mathbf{U}_i\}$  of  $\mathbf{X}$ , together with an absolute Frobenius lifting  $F_{\mathbf{U}_i}$  on each  $\mathbf{U}_i$ . By modulo  $p$ , the covering induces an affine covering  $\{U_i\}$  for  $X$ . We show first a special case of the theorem.

**Proposition 3.2.** *There is a one to one correspondence between the Faltings category  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$  and the category  $\mathcal{HB}_{n,(0,1)}(X/k)$ .*

Let  $(H, \nabla, \text{Fil}, \Phi)$  be an object in  $\mathcal{MF}$ . Put  $(E, \theta) := \text{Gr}_{\text{Fil}}(H, \nabla)$ . The following lemma gives a functor  $\mathcal{GR}$  from the category  $\mathcal{MF}$  to the category  $\mathcal{HB}_{(0,1)}$ .

**Lemma 3.3.** *There is a filtration  $\text{Fil}_{\text{exp}}$  on  $C_0^{-1}(E, \theta)$  together with an isomorphism of Higgs bundles*

$$\phi_{\text{exp}} : \text{Gr}_{\text{Fil}_{\text{exp}}}(C_0^{-1}(E, \theta)) \cong (E, \theta),$$

*which is induced by the Hodge filtration  $\text{Fil}$  and the relative Frobenius  $\Phi$ .*

*Proof.* By Proposition 5 [9], we showed that the relative Frobenius induces a global isomorphism of flat bundles

$$\tilde{\Phi} : C_0^{-1}(E, \theta) \cong (H, \nabla).$$

So we define  $\text{Fil}_{\text{exp}}$  on  $C_0^{-1}(E, \theta)$  to be the inverse image of  $\text{Fil}$  on  $H$  by  $\tilde{\Phi}$ . It induces tautologically an isomorphism of Higgs bundles

$$\phi_{\text{exp}} = \text{Gr}(\tilde{\Phi}) : \text{Gr}_{\text{Fil}_{\text{exp}}}(C_0^{-1}(E, \theta)) \cong (E, \theta).$$

□

Next, we show that the functor  $C_0^{-1}$  induces a functor in the opposite direction. Given an object  $(E, \theta, \text{Fil}, \phi) \in \mathcal{HB}_{(0,1)}$ , it is clear to define the triple

$$(H, \nabla, \text{Fil}) = (C_0^{-1}(E, \theta), \text{Fil}).$$

What remains is to produce a relative Frobenius  $\Phi$  from the  $\phi$ . Following Faltings [1] Ch. II. d), it suffices to give for each pair  $(\mathbf{U}_i, F_{\mathbf{U}_i})$  an  $\mathcal{O}_{U_i}$ -morphism

$$\Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})} : F_{U_i}^* \text{Gr}_{\text{Fil}} H|_{U_i} \rightarrow H|_{U_i}$$

satisfying

- (1) strong  $p$ -divisibility, that is,  $\Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})}$  is an isomorphism,
- (2) horizontal property,
- (3) over each  $U_i \cap U_j$ ,  $\Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})}$  and  $\Phi_{(\mathbf{U}_j, F_{\mathbf{U}_j})}$  are related via the Taylor formula.

Recall [9] that over each  $U_i$  we have the identification (chart)

$$\alpha_i := \alpha_{(\mathbf{U}_i, F_{\mathbf{U}_i})} : (F_{U_i}^* E|_{U_i}, d + \frac{dF_{\mathbf{U}_i}}{p} F_{U_i}^* \theta|_{U_i}) \cong C_0^{-1}(E, \theta)|_{U_i}.$$

We define  $\Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})}$  to be the composite

$$F_{U_i}^* \text{Gr}_{\text{Fil}} H|_{U_i} \xrightarrow{F_{U_i}^* \phi} F_{U_i}^* E|_{U_i} \xrightarrow{\alpha_i} C_0^{-1}(E, \theta)|_{U_i} = H|_{U_i}.$$

By construction,  $\Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})}$  is strongly  $p$ -divisible. By Proposition 5 loc. cit., the transition function between  $\alpha_i$  and  $\alpha_j$  is given by the Taylor formula. It follows that  $\Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})}$  and  $\Phi_{(\mathbf{U}_j, F_{\mathbf{U}_j})}$  are interrelated by the Taylor formula.

**Lemma 3.4.** *Each  $\Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})}$  is horizontal with respect to  $\nabla$ .*

*Proof.* Put  $\tilde{H} = Gr_{Fil}H$ ,  $\theta' = Gr_{Fil}\nabla$ ,  $\Phi_i = \Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})}$  and  $F_0$  the absolute Frobenius over  $U_i$ . Following Faltings [1] Ch. II. d), it is to show the following commutative diagram

$$\begin{array}{ccc} F_0^* \tilde{H}|_{U_i} & \xrightarrow{\Phi_i} & H|_{U_i} \\ F_{\mathbf{U}_i}^* \nabla \downarrow & & \nabla \downarrow \\ F_0^* \tilde{H}|_{U_i} \otimes \Omega_{U_i} & \xrightarrow{\Phi_i \otimes id} & H|_{U_i} \otimes \Omega_{U_i}. \end{array}$$

Here  $F_{\mathbf{U}_i}^* \nabla$  is just the composite of

$$F_0^* \tilde{H}|_{U_i} \xrightarrow{F_0^* \theta'} F_0^* \tilde{H}|_{U_i} \otimes F_0^* \Omega_{U_i} \xrightarrow{id \otimes \frac{dF_{\mathbf{U}_i}}{p}} F_0^* \tilde{H}|_{U_i} \otimes \Omega_{U_i}.$$

Via the identification  $\alpha_i$ , it is reduced to show the following diagram commutes:

$$\begin{array}{ccc} F_0^* \tilde{H}|_{U_i} & \xrightarrow{F_0^* \phi} & F_0^* E|_{U_i} \\ F_{\mathbf{U}_i}^* \nabla \downarrow & & \frac{dF_{\mathbf{U}_i}}{p} F_0^* \theta \downarrow \\ F_0^* \tilde{H}|_{U_i} \otimes \Omega_{U_i} & \xrightarrow{F_0^* \phi \otimes id} & F_0^* E|_{U_i} \otimes \Omega_{U_i}. \end{array}$$

As  $\phi$  is a morphism of Higgs bundles, one has the following commutative diagram:

$$\begin{array}{ccc} \tilde{H}|_{U_i} & \xrightarrow{\phi} & E|_{U_i} \\ \theta' \downarrow & & \downarrow \theta \\ \tilde{H}|_{U_i} \otimes \Omega_{U_i} & \xrightarrow{\phi \otimes id} & E|_{U_i} \otimes \Omega_{U_i}. \end{array}$$

The pull-back via  $F_0^*$  of the above diagram yields the next commutative diagram

$$\begin{array}{ccccc} F_0^* \tilde{H}|_{U_i} & \xrightarrow{F_0^* \phi} & F_0^* E|_{U_i} & & \\ F_0^* \theta' \downarrow & & F_0^* \theta \downarrow & \searrow \frac{dF_{\mathbf{U}_i}}{p} F_0^* \theta & \\ F_0^* \tilde{H}|_{U_i} \otimes F_0^* \Omega_{U_i} & \xrightarrow{F_0^* \phi \otimes id} & F_0^* E|_{U_i} \otimes F_0^* \Omega_{U_i} & \xrightarrow{id \otimes \frac{dF_{\mathbf{U}_i}}{p}} & F_0^* E|_{U_i} \otimes \Omega_{U_i}. \\ & \searrow F_0^* \phi \otimes \frac{dF_{\mathbf{U}_i}}{p} & & & \end{array}$$

The commutativity of the second diagram follows now from that of the last diagram.  $\square$

The above lemma provides us with the functor  $\mathcal{C}_0^{-1}$  in the opposite direction. Now we can prove Proposition 3.2.

*Proof.* The equivalence of categories follows by providing natural isomorphisms of functors:

$$\mathcal{GR} \circ \mathcal{C}_0^{-1} \cong Id, \quad \mathcal{C}_0^{-1} \circ \mathcal{GR} \cong Id.$$

We define first a natural isomorphism  $\mathcal{A}$  from  $\mathcal{C}_0^{-1} \circ \mathcal{GR}$  to  $Id$ : for  $(H, \nabla, Fil, \Phi) \in \mathcal{MF}$ , put

$$(E, \theta, Fil, \phi) = \mathcal{GR}(H, \nabla, Fil, \Phi), \quad (H', \nabla', Fil', \Phi') = \mathcal{C}_0^{-1}(E, \theta, Fil, \phi).$$

Then one verifies that the map

$$\tilde{\Phi} : (H', \nabla') = \mathcal{C}_0^{-1} \circ Gr_{Fil}(H, \nabla) \cong (H, \nabla)$$

gives an isomorphism from  $(H', \nabla', Fil', \Phi')$  to  $(H, \nabla, Fil, \Phi)$  in the category  $\mathcal{MF}$ . We call it  $\mathcal{A}(H, \nabla, Fil, \Phi)$ . It is straightforward to verify that  $\mathcal{A}$  is indeed a transformation. Conversely, a natural isomorphism  $\mathcal{B}$  from  $\mathcal{GR} \circ \mathcal{C}_0^{-1}$  to  $Id$  is given as follows: for  $(E, \theta, Fil, \phi)$ , put

$$(H, \nabla, Fil, \Phi) = \mathcal{C}_0^{-1}(E, \theta, Fil, \phi) \quad (E', \theta', Fil', \phi') = \mathcal{GR}(H, \nabla, Fil, \Phi).$$

Then  $\phi : Gr_{Fil} \circ \mathcal{C}_0^{-1}(E, \theta) \cong (E, \theta)$  induces an isomorphism from  $(E', \theta', Fil', \phi')$  to  $(E, \theta, Fil, \phi)$  in  $\mathcal{HB}_{(0,1)}$ , which we define to be  $\mathcal{B}(E, \theta, Fil, \phi)$ . It is direct to check that  $\mathcal{B}$  is a natural isomorphism.  $\square$

Before moving to the proof of Theorem 3.1 in general, we shall introduce an intermediate category, the category of periodic Higgs-de Rham sequences of type  $(0, 1)$  with endomorphism structure  $\mathbb{F}_{p^f}$ : an object is a five tuple  $(E, \theta, Fil, \phi, \iota)$ , where  $(E, \theta, Fil, \phi)$  is object in  $\mathcal{HB}_{(0,1)}$  and  $\iota : \mathbb{F}_{p^f} \hookrightarrow \text{End}_{\mathcal{HB}_{(0,1)}}(E, \theta, Fil, \phi)$  is an embedding of  $\mathbb{F}_p$ -algebras. We denote this category by  $\mathcal{HB}_f$ . A direct consequence of Proposition 3.2 is the following

**Corollary 3.5.** *The category  $\mathcal{MF}_{[0,n],f}^\nabla(\mathbf{X}/W)$  is equivalent to the category  $\mathcal{HB}_f$  of Higgs-de Rham sequences of type  $(0, 1)$  with endomorphism structure  $\mathbb{F}_{p^f}$ .*

Corollary 3.5 and the following proposition finish the proof of Theorem 3.1.

**Proposition 3.6.** *There is a one to one correspondence between the category  $\mathcal{HB}_{(0,f)}$  of periodic Higgs-de Rham sequences of type  $(0, f)$  and the category  $\mathcal{HB}_f$  of periodic Higgs-de Rham sequences of type  $(0, 1)$  with endomorphism structure  $\mathbb{F}_{p^f}$ .*

We start with an object  $(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi)$  in  $\mathcal{HB}_{(0,f)}$ . Put

$$(G, \eta) := \bigoplus_{i=0}^{f-1} (E_i, \theta_i)$$

with  $(E_0, \theta_0) = (E, \theta)$ . As the functor  $\mathcal{C}_0^{-1}$  is compatible with direct sum, one has the identification

$$\mathcal{C}_0^{-1}(G, \eta) = \bigoplus_{i=0}^{f-1} \mathcal{C}_0^{-1}(E_i, \theta_i).$$

We equip the filtration  $Fil$  on  $\mathcal{C}_0^{-1}(G, \eta)$  by  $\bigoplus_{i=0}^{f-1} Fil_i$  via the above identification. Also  $\phi$  induces a natural isomorphism of Higgs bundles  $\tilde{\phi} : Gr_{Fil} \mathcal{C}_0^{-1}(G, \eta) \cong (G, \eta)$  as follows: as

$$Gr_{Fil} \mathcal{C}_0^{-1}(G, \eta) = \bigoplus_{i=0}^{r-1} Gr_{Fil_i} \mathcal{C}_0^{-1}(E_i, \theta_i),$$

we require that  $\tilde{\phi}$  maps the factor  $Gr_{Fil_i}(E_i, \theta_i)$  identically to the factor  $(E_{i+1}, \theta_{i+1})$  for  $0 \leq i \leq f-2$  (assume  $f \geq 2$  to avoid the trivial case) and the last factor  $Gr_{Fil_{f-1}}(E_{f-1}, \theta_{f-1})$  isomorphically to  $(E_0, \theta_0)$  via  $\phi$ . Thus the so constructed four tuple  $(G, \eta, Fil, \tilde{\phi})$  is an object in  $\mathcal{HB}_{(0,1)}$ .

**Lemma 3.7.** *For an object  $(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi)$  in  $\mathcal{HB}_{(0,f)}$ , there is a natural embedding of  $\mathbb{F}_p$ -algebras*

$$\iota : \mathbb{F}_{p^r} \rightarrow \text{End}_{\mathcal{HB}_{(0,1)}}(G, \eta, Fil, \tilde{\phi}).$$

*Thus the extended tuple  $(G, \eta, Fil, \tilde{\phi}, \iota)$  is an object in  $\mathcal{HB}_f$ .*

*Proof.* Without loss of generality, we assume  $f = 2$ . Choose a primitive element  $\xi$  in  $\mathbb{F}_{p^r}|\mathbb{F}_p$  once and for all. To define the embedding  $\iota$ , it suffices to specify the image  $s := \iota(\xi)$ , which is defined as follows: write  $(G, \eta) = (E_0, \theta_0) \oplus (E_1, \theta_1)$ . Then  $s = m_\xi \oplus m_{\xi^p}$ , where  $m_{\xi^{p^i}}, i = 0, 1$  is the multiplication map by  $\xi^{p^i}$ . It defines an endomorphism of  $(G, \eta)$  and preserves  $Fil$  on  $C_0^{-1}(G, \eta)$ . Write  $(Gr_{Fil} \circ C_0^{-1})(s)$  to be the induced endomorphism of  $Gr_{Fil}C_0^{-1}(G, \eta)$ . It remains to verify the commutativity

$$\tilde{\phi} \circ s = (Gr_{Fil} \circ C_0^{-1})(s) \circ \tilde{\phi}.$$

In terms of a local basis, it boils down to the equation

$$\begin{pmatrix} 0 & 1 \\ \phi & 0 \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & \xi^p \end{pmatrix} = \begin{pmatrix} \xi^p & 0 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \phi & 0 \end{pmatrix},$$

which is clear.  $\square$

Conversely, given an object  $(G, \eta, Fil, \phi, \iota)$  in the category  $\mathcal{HB}_f$ , we can associate it an object in  $\mathcal{HB}_{(0,f)}$  as follows: the endomorphism  $\iota(\xi)$  decomposes  $(G, \eta)$  into eigenspaces:

$$(G, \eta) = \bigoplus_{i=0}^{f-1} (G_i, \eta_i),$$

where  $(G_i, \eta_i)$  is the eigenspace to the eigenvalue  $\xi^{p^i}$ . The isomorphism  $C_0^{-1}(\iota(\xi))$  induces the eigen-decomposition of the de Rham bundle as well:

$$(C_0^{-1}(G, \eta), Fil) = \bigoplus_{i=0}^{f-1} (C_0^{-1}(G_i, \eta_i), Fil_i).$$

Under the decomposition, the isomorphism  $\phi : Gr_{Fil}C_0^{-1}(G, \eta) \cong (G, \eta)$  decomposes into  $\bigoplus_{i=0}^{f-1} \phi_i$  such that

$$\phi_i : Gr_{Fil_i}C_0^{-1}(G_i, \eta_i) \cong (G_{i+1 \bmod f}, \theta_{i+1 \bmod f}).$$

Put  $(E, \theta) = (G_0, \theta_0)$ .

**Lemma 3.8.** *The filtrations  $\{Fil_i\}_s$  and isomorphisms of Higgs bundles  $\{\phi_i\}_s$  induce inductively the filtration  $\widetilde{Fil}_i$  on  $C_0^{-1}(E_i, \theta_i), i = 0, \dots, f-1$  and the isomorphism of Higgs bundles*

$$\tilde{\phi} : Gr_{\widetilde{Fil}_{f-1}}(E_{f-1}, \theta_{f-1}) \cong (E, \theta).$$

*Thus the extended tuple  $(E, \theta, \widetilde{Fil}_0, \dots, \widetilde{Fil}_{f-1}, \tilde{\phi})$  is an object in  $\mathcal{HB}_{(0,f)}$ .*

*Proof.* Again we shall assume  $f = 2$ . The filtration  $\widetilde{Fil}_0$  on  $C_0^{-1}(E_0, \theta_0)$  is just  $Fil_0$ . Via the isomorphism

$$C_0^{-1}(\phi_0) : C_0^{-1}Gr_{Fil_0}C_0^{-1}(G_0, \eta_0) \cong C_0^{-1}(G_1, \eta_1),$$

we obtain the filtration  $\widetilde{Fil}_1$  on  $C_0^{-1}(E_1, \theta_1)$  from the  $Fil_1$ . Finally we define  $\tilde{\phi}$  to be the composite:

$$Gr_{\widetilde{Fil}_1}(E_1, \theta_1) = Gr_{\widetilde{Fil}_1}C_0^{-1}Gr_{\widetilde{Fil}_0}C_0^{-1}(E, \theta) \xrightarrow{Gr_{\widetilde{Fil}_1}C_0^{-1}(\phi_0)} Gr_{\widetilde{Fil}_1}C_0^{-1}(G_1, \eta_1) \xrightarrow{\phi_1} (E, \theta).$$

$\square$

We come to the proof of Proposition 3.6.

*Proof.* Note first that Lemma 3.7 gives us a functor  $\mathcal{E}$  from  $\mathcal{HB}_{(0,f)}$  to  $\mathcal{HB}_f$ , while Lemma 3.8 a functor  $\mathcal{F}$  in the opposite direction. We show that they give an equivalence of categories. It is direct to see that

$$\mathcal{F} \circ \mathcal{E} = Id.$$

So it remains to give a natural isomorphism  $\tau$  between  $\mathcal{E} \circ \mathcal{F}$  and  $Id$ . Again we assume that  $f = 2$  in the following argument. For  $(E, \theta, Fil, \phi, \iota)$ , put

$$\mathcal{F}\{(E, \theta, Fil, \phi, \iota)\} = (G, \eta, Fil_0, Fil_1, \tilde{\phi}), \quad \mathcal{E}(G, \eta, Fil_0, Fil_1, \tilde{\phi}) = (E', \theta', Fil', \phi', \iota').$$

Notice that  $(E', \theta') = (G, \eta) \oplus Gr_{Fil_0} C_0^{-1}(G, \eta)$ , we define an isomorphism of Higgs bundles by

$$Id \oplus \phi_0 : (E', \theta') = (G, \eta) \oplus Gr_{Fil_0} C_0^{-1}(G, \eta) \cong (E_0, \theta_0) \oplus (E_1, \theta_1) = (E, \theta).$$

It is easy to check that the above isomorphism gives an isomorphism  $\tau(E, \theta, Fil, \phi, \iota)$  in the category  $\mathcal{HB}_f$ . The functorial property of  $\tau$  is easily verified.  $\square$

Faltings showed that the (contravariant) functor  $\mathbf{D} [1]$  from  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$  to the category of continuous  $\mathbb{F}_p$ -representations of  $\pi_1(\mathbf{X}^0)$  is fully faithful. The image is closed under subobject and quotient, and its object is called dual crystalline sheaf. In our paper we take the dual of  $\mathbf{D}$  (cf. page 43 loc. cit.) without changing the notation. A crystalline  $\mathbb{F}_{p^f}$ -representation is a crystalline  $\mathbb{F}_p$ -representation  $\mathbb{V}$  with an embedding of  $\mathbb{F}_p$ -algebras  $\mathbb{F}_{p^f} \hookrightarrow \text{End}_{\pi_1(\mathbf{X}^0)}(\mathbb{V})$ .

**Corollary 3.9.** *There is an equivalence of categories between the category of crystalline  $\mathbb{F}_{p^f}$ -representations of  $\pi_1(\mathbf{X}^0)$  and the category of periodic Higgs-de Rham sequences of type  $(0, f)$ .*

*Proof.* Under the functor  $\mathbf{D}$ , an  $\mathbb{F}_{p^f}$ -endomorphism structure on an object of  $\mathcal{MF}$  is mapped to an  $\mathbb{F}_{p^f}$ -endomorphism structure on the corresponding  $\mathbb{F}_p$ -representation, and vice versa. The result is then a direct consequence of Theorem 3.1.  $\square$

Let  $\rho$  be a crystalline  $\mathbb{F}_{p^f}$ -representation of  $\pi_1(\mathbf{X}^0)$ , and  $(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi)$  the corresponding periodic Higgs-de Rham sequence of type  $(0, f)$ . For

$$(E_f, \theta_f) = Gr_{Fil_{f-1}}(H_{f-1}, \nabla_{f-1}),$$

$C_0^{-1}(\phi)$  induces the pull-back filtration  $C_0^{-1}(\phi)^* Fil_0$  on  $C_0^{-1}(E_f, \theta_f)$  and an isomorphism of Higgs bundles  $GrC_0^{-1}(\phi)$  on the gradings. It is easy to check that

$$(E_1, \theta_1, Fil_1, \dots, Fil_{f-1}, C_0^{-1}(\phi)^* Fil_0, GrC_0^{-1}(\phi))$$

is an object in  $\mathcal{HB}_{(0,f)}$ , which is called the *shift* of  $(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi)$ . For any multiple  $lf, l \geq 1$ , we can lengthen  $(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi)$  to an object of  $\mathcal{HB}_{(0,lf)}$ : as above, we can inductively define the induced filtration on  $(H_j, \nabla_j), j \leq j \leq lf - 1$  from  $Fil_i$ s via  $\phi$ . One has the induced isomorphism of Higgs bundles  $(GrC_0^{-1})^{l'f}(\phi) : (E_{(\nu+1)f}, \theta_{(\nu+1)f}) \cong (E_{\nu f}, \theta_{\nu f}), 0 \leq l' \leq l - 1$ . The isomorphism  $\phi_l : (E_{lf}, \theta_{lf}) \cong (E_0, \theta_0)$  is defined to be the composite of them. The obtained object  $(E, \theta, Fil_0, \dots, Fil_{lf-1}, \phi_l)$  is called the  $l$ -th *lengthening* of  $(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi)$ . The following result is obvious from the construction of the above correspondence.

**Proposition 3.10.** *Let  $\rho$  and  $(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi)$  be as above. Then the followings are true:*

- (i) *The shift of  $(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi)$  corresponds to  $\rho^\sigma = \rho \otimes_{\mathbb{F}_{p^f}, \sigma} \mathbb{F}_{p^f}$ , the  $\sigma$ -conjugation of  $\rho$ . Here  $\sigma \in \text{Gal}(\mathbb{F}_{p^f} | \mathbb{F}_p)$  is the Frobenius element.*

- (ii) For  $l \in \mathbb{N}$ , the  $l$ -th lengthening of  $(E, \theta, \text{Fil}_0, \dots, \text{Fil}_{f-1}, \phi)$  corresponds to the base extension  $\rho \otimes_{\mathbb{F}_{p^f}} \mathbb{F}_{p^{lf}}$ .

We remind also the reader of the following result.

**Corollary 3.11.** *Periodic Higgs bundles are locally free.*

*Proof.* Let  $(E, \theta)$  be a periodic Higgs bundle. Then a Higgs-de Rham sequence for it gives an object in the category  $\mathcal{HB}_{(0,f)}$  for a certain  $f$ . Let  $(H, \nabla, \text{Fil}, \Phi, \iota)$  be the corresponding object in  $\mathcal{MF}_f$ . The proof of Theorem 2.1 [1] (cf. page 32 loc. cit.) says that  $\text{Fil}$  is a filtration of locally free subsheaves of  $H$  and the grading  $\text{Gr}_{\text{Fil}} H$  is also locally free. It follows immediately that  $(E, \theta)$  is locally free.  $\square$

#### 4. QUASI-PERIODIC HIGGS BUNDLES

A quasi-periodic Higgs-de Rham sequence of type  $(e, f)$  is a tuple

$$(E, \theta, \text{Fil}_0, \dots, \text{Fil}_{e+f-1}, \phi),$$

where  $\phi$  is an isomorphism of Higgs bundles

$$\phi : \text{Gr}_{\text{Fil}_{e+f-1}}(H_{e+f-1}, \nabla_{e+f-1}) \cong (E_e, \theta_e).$$

It follows from Corollary 3.11 that the Higgs bundles  $(E_i, \theta_i)$ ,  $e \leq i \leq e + f - 1$  are locally free. They form the category  $\mathcal{HB}_{n,(e,f)}(X/k)$ .

We are going to associate a quasi-periodic Higgs-de Rham sequence of type  $(e, f)$  with an object in a Faltings category. We recall first the strict  $p$ -torsion category  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}_V/R_V)$ , which is based on the category introduced by Faltings in §3-§4 [2]. For  $V$  a totally ramified extension of  $W(k)$ , Faltings §2 [2] introduced the base ring  $R_V$  as follows: a uniformizer  $\pi$  of  $V$  has the minimal polynomial

$$f(T) = T^e + \sum_{0 < i < e} a_i T^i \in W[T].$$

It defines the  $W$ -algebra morphism  $W[[T]] \rightarrow V, T \mapsto \pi$  and  $R_V$  is defined to be the PD-hull of  $V$ . One has an excellent lifting  $X/k$  over  $R_V$ , that is, one takes  $\mathbf{X} \times_W R_V$ , the base change of  $\mathbf{X}/W$  to  $R_V$ . Put  $\mathcal{X} = \mathbf{X} \times_W R_V/p = X \times_k R_V/p$ . It depends only on the ramification index  $e$  of  $V$ , not on  $V$  itself. The sheaf of  $k$ -algebras  $\mathcal{O}_{\mathcal{X}}$  admits a natural filtration  $\text{Fil}_{\mathcal{O}_{\mathcal{X}}}$ . The composite of the natural maps

$$k = W/p \rightarrow R_V/p \xrightarrow{T \mapsto 0} k$$

is the identity. It induces the commutative diagram of  $k$ -schemes

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathcal{X} \\ & \searrow id & \downarrow \lambda \\ & & X. \end{array}$$

An object of the category  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}_V/R_V)$  is a four tuple  $(H, \nabla, \text{Fil}, \Phi)$ , where  $(H, \text{Fil})$  is a locally filtered-free  $\mathcal{O}_{\mathcal{X}}$ -module of finite rank, with a local basis consisting of homogenous elements of degrees between 0 and  $n$ ,  $\nabla : H \rightarrow H \otimes \Omega_{\mathcal{X}/k}$  an integrable connection satisfying the Griffiths transversality, the relative Frobenius  $\Phi$  is strongly  $p$ -divisible (i.e.  $\Phi$  locally over  $\mathcal{U}_i \subset \mathcal{X}$  induces an isomorphism  $F_{\mathcal{U}_i}^* \text{Gr}_{\text{Fil}}^n H \cong H|_{\mathcal{U}_i}$ ) and horizontal with respect to  $\nabla$ .

**Lemma 4.1.** *The morphism  $\lambda$  induces a functor  $\lambda^*$  from  $\mathcal{HB}_{(e,f)}$  to  $\mathcal{MF}_{[0,n],f}^\nabla(\mathbf{X}_V/R_V)$  and the morphism  $\mu$  a functor  $\mu^*$  from  $\mathcal{MF}_{[0,n],f}^\nabla(\mathbf{X}_V/R_V)$  to the category  $\mathcal{HB}_{(0,f)}$ .*

*Proof.* For  $(E, \theta, \text{Fil}_0, \dots, \text{Fil}_{e+f-1}, \phi)$ , we take  $(E', \theta') = \bigoplus_{i=0}^{f-1} (E_i, \theta_i)$ . Then  $\text{Fil}_i$  and  $\phi$  induces naturally an object  $(E', \theta', \text{Fil}'_0, \dots, \text{Fil}'_e, \phi')$  in  $\mathcal{HB}_{(e,1)}$ . Thus it suffices to show the above statement for  $f = 1$ .

Put  $H = \lambda^* H_e$ ,  $\nabla = \lambda^* \nabla_e$  and  $\text{Fil} = \text{Fil}_{\mathcal{O}_X} \otimes \lambda^* \text{Fil}_e$ . Note that one has a natural isomorphism of  $\mathcal{O}_X$ -modules  $F_{\mathcal{U}_i}^* \text{Gr}_{\text{Fil}}^n H \cong \lambda^* F_{U_i}^* \text{Gr}_{\text{Fil}_e} H_e$ . We define the relative Frobenius  $\Phi$  on  $H$  via the above isomorphism composed with  $\lambda^* \Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})}$ , where  $\Phi_{(\mathbf{U}_i, F_{\mathbf{U}_i})} : F_{U_i}^* \text{Gr}_{\text{Fil}_e} H_e \rightarrow H_e|_{U_i}$  appeared in the paragraph before Lemma 3.4. This gives us the functor  $\lambda^*$  from  $\mathcal{HB}_{(e,1)}$  to  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}_V/R_V)$ . Conversely, given an object  $(H, \nabla, \text{Fil}, \Phi) \in \mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}_V/R_V)$ , the tuple  $(\mu^* H, \mu^* \nabla, \mu^* \text{Fil}, \mu^* \Phi)$  is naturally an object in  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$ : over  $\mathcal{U}_i$ ,  $\Phi$  gives an isomorphism  $F_{\mathcal{U}_i}^* \text{Gr}_{\text{Fil}}^n H \cong H_{\mathcal{U}_i}$ . Pulling back the isomorphism via  $\mu$ , we get  $F_{U_i}^* \mu^* \text{Gr}_{\text{Fil}}^n H \cong \mu^* H|_{U_i}$ . As there is a natural  $\mathcal{O}_X$ -modules isomorphism  $\text{Gr}_{\mu^* \text{Fil}} \mu^* H \cong \mu^* \text{Gr}_{\text{Fil}}^n H$ , we have an isomorphism  $F_{U_i}^* \text{Gr}_{\mu^* \text{Fil}} \mu^* H|_{U_i} \cong \mu^* H|_{U_i}$ , which shows that  $\mu^* \Phi$  is indeed a relative Frobenius. We define  $\mu^*(H, \nabla, \text{Fil}, \Phi) \in \mathcal{HB}_{(0,1)}$  to be the object associated to  $(\mu^* H, \mu^* \nabla, \mu^* \text{Fil}, \mu^* \Phi)$ .  $\square$

**Corollary 4.2.** *There is a functor from the category of quasi-periodic Higgs-de Rham sequences of type  $(e, f)$  to the category of crystalline representations of  $\pi_1(\mathbf{X}'^0)$  into  $\text{GL}(\mathbb{F}_{p^f})$ , where  $\mathbf{X}'^0$  is the generic fiber of  $\mathbf{X}' := \mathbf{X} \times_W \mathcal{O}_K$  for a totally ramified extension  $\text{Frac}(W) \subset K$  with ramification index  $e$ . There is also a functor in the converse direction.*

*Proof.* The first part follows from the above functor  $\lambda^*$  and the proof of Theorem 5. i) [2]. To provide a functor in the opposite direction, we use the functor  $\mu^*$  together with choosing an additional embedding of the category  $\mathcal{HB}_{(0,f)}$  into  $\mathcal{HB}_{(e,f)}$ . This can be done as follows: for an object  $(E, \theta, \text{Fil}_0, \dots, \text{Fil}_{f-1}, \phi) \in \mathcal{HB}_{(0,f)}$ , let  $l \in \mathbb{N}$  be the minimal number with  $e \leq lf$ . Then there is a unique object  $(E', \theta', \text{Fil}'_0, \dots, \text{Fil}'_{e+f-1}, \phi')$  in  $\mathcal{HB}_{(e,f)}$  obtained from its  $l+1$ -th lengthening which satisfies the equality

$$(E'_i, \theta'_i) = (E_{lf-e+i}, \theta_{lf-e+i}), 0 \leq i \leq e+f.$$

$\square$

## 5. APPLICATIONS

Given a periodic Higgs-de Rham sequence

$$\begin{array}{ccccc} & (H_0, \nabla_0) & & (H_1, \nabla_1) & \\ C_0^{-1} \nearrow & & \text{Gr}_{\text{Fil}_0} \searrow & C_0^{-1} \nearrow & \text{Gr}_{\text{Fil}_1} \searrow \\ (E_0, \theta_0) & & (E_1, \theta_1) & & \dots \end{array}$$

we make the following observation:

**Lemma 5.1.** *If  $(E, \theta) = (E_0, \theta_0)$  is Higgs stable, then there is a unique periodic Higgs-de Rham sequence for  $(E, \theta)$  up to isomorphism.*

*Proof.* Let  $f \in \mathbb{N}$  be the period of the sequence. Thus there is an isomorphism  $\phi : (E_f, \theta_f) \cong (E_0, \theta_0)$  such that the tuple  $(E, \theta, \text{Fil}_0, \dots, \text{Fil}_{f-1}, \phi)$  makes an object in

$\mathcal{HB}_{(0,f)}$ . We show that the datum  $Fil_i, 0 \leq i \leq f-1$  and  $\phi$  are uniquely determined up to isomorphism. By Theorem 3.1, there is a corresponding object

$$(H, Fil, \nabla, \Phi, \iota) \in \mathcal{MF}_f$$

satisfying  $Gr_{Fil}(H, \nabla) = \bigoplus_{i=1}^f (E_i, \theta_i)$ . Because it holds that

$$(Gr_{Fil} \circ C_0^{-1})^i(E_f, \theta_f) = (E_i, \theta_i), 1 \leq i \leq f-1,$$

each  $(E_i, \theta_i)$  is also Higgs stable by Corollary 4.4 [11]. Now we show inductively that  $Fil_i$  is unique. This is because of the fact that there is a unique filtration on a flat bundle which satisfies the Griffiths transversality and its grading is Higgs stable. Now we consider  $\phi$ . For another choice  $\varphi$ , one notes that  $\varphi \circ \phi^{-1}$  is an automorphism of  $(E, \theta)$ . As it is stable, one must have  $\varphi = \lambda\phi$  for a nonzero  $\lambda$  in  $k$ . It is easy to see there is an isomorphism in  $\mathcal{HB}_{(0,f)}$ :

$$(E, \theta, Fil_0, \dots, Fil_{f-1}, \phi) \cong (E, \theta, Fil_0, \dots, Fil_{f-1}, \lambda\phi).$$

□

Because of the above lemma, the period of a periodic Higgs stable bundle is well defined. We make then the following statement.

**Corollary 5.2.** *Under the equivalence of categories in Corollary 3.9, there is one to one correspondence between the isomorphism classes of irreducible crystalline  $\mathbb{F}_p$ -representations of  $\pi_1(\mathbf{X}^0)$  and the isomorphism classes of periodic Higgs stable bundles of period  $f$ .*

The first examples of periodic Higgs stable bundles are the rank two Higgs subbundles of uniformizing type arising from the study of the Higgs bundle of a universal family of abelian varieties over the good reduction of a Shimura curve of PEL type (see [12]). In that case, one 'sees' the corresponding representations because of the existence of extra endomorphisms in the universal family. The above result gives a vast generalization of this primitive example.

When a periodic Higgs bundle  $(E, \theta)$  is only Higgs semistable, the above uniqueness statement is no longer true. We shall make the following

**Assumption 5.3.** For each  $0 \leq i \leq f-1$ , the filtration  $Fil_i$  on  $H_i$  is preserved by any automorphism of  $(H_i, \nabla_i)$ .

An isomorphism  $\varphi : (E_f, \theta_f) \cong (E_0, \theta_0)$  induces

$$(GrC_0^{-1})^{nf}(\varphi) : (E_{(n+1)f}, \theta_{(n+1)f}) \cong (E_{nf}, \theta_{nf}).$$

For  $-1 \leq i < j$ , we define

$$\varphi_{j,i} = (GrC_0^{-1})^{(i+1)f}(\varphi) \circ \dots \circ (GrC_0^{-1})^{jf}(\varphi) : (E_{(j+1)f}, \theta_{(j+1)f}) \cong (E_{(i+1)f}, \theta_{(i+1)f}).$$

For  $i = -1$  put  $\varphi_j = \varphi_{j,-1}$ .

**Lemma 5.4.** *For any two isomorphisms  $\varphi, \phi : (E_f, \theta_f) \cong (E_0, \theta_0)$ , there exists a pair  $(i, j)$  with  $0 \leq i < j$  such that  $\phi_{j,i} \circ \varphi_{j,i}^{-1} = id$ .*

*Proof.* If we denote  $\tau_s = \phi_s \circ \varphi_s^{-1}$ , then  $\tau_s$  is an automorphism of  $(E_0, \theta_0)$ . Moreover, each element in the set  $\{\tau_s\}_{s \in \mathbb{N}}$  is defined over the same finite field in  $k$ . As this is a finite set, there are  $j > i \geq 0$  such that  $\tau_j = \tau_i$ . So the lemma follows. □

**Proposition 5.5.** *Assume 5.3. Let  $(i, j)$  be a pair given by Lemma 5.4 for two given isomorphisms  $\varphi, \phi : (E_f, \theta_f) \cong (E_0, \theta_0)$ . Then there is an isomorphism in  $\mathcal{HB}_{(0, (j-i)f)}$ :*

$$(E, \theta, \text{Fil}_0, \dots, \text{Fil}_{f-1}, \varphi_{j-i-1}) \cong (E, \theta, \text{Fil}_0, \dots, \text{Fil}_{f-1}, \phi_{j-i-1}).$$

*Proof.* Put  $\beta = \phi_i \circ \varphi_i^{-1} : (E_0, \theta_0) \cong (E_0, \theta_0)$ . We shall check that it induces an isomorphism in  $\mathcal{HB}_{(0, (j-i)f)}$ . By Assumption 5.3,  $C_0^{-1}(\text{Gr}C_0^{-1})^m(\beta)$  for  $m \geq 0$  always respects the filtrations. We need only to check that  $\beta$  is compatible with  $\phi_{j-i-1}$  as well as  $\varphi_{j-i-1}$ . So it suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} E_{(j-i)f} & \xrightarrow{\varphi_{j-i-1}} & E_0 \\ \downarrow \varphi_{j,j-i-1}^{-1} & & \downarrow \varphi_i^{-1} \\ E_{(j+1)f} & & E_{(i+1)f} \\ \downarrow \phi_{j,j-i-1} & & \downarrow \phi_i \\ E_{(j-i)f} & \xrightarrow{\phi_{j-i-1}} & E_0 \end{array}$$

And it suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} E_{(j-i)f} & \xleftarrow{\varphi_{j-i-1}^{-1}} & E_0 \\ \downarrow \varphi_{j,j-i-1}^{-1} & & \downarrow \varphi_i^{-1} \\ E_{(j+1)f} & & E_{(i+1)f} \\ \downarrow \phi_{j,j-i-1} & & \downarrow \phi_i \\ E_{(j-i)f} & \xrightarrow{\phi_{j-i-1}} & E_0 \end{array}$$

In the above diagram, the anti-clockwise direction is

$$\phi_{j-i-1} \circ \phi_{j,j-i-1} \circ \varphi_{j,j-i-1}^{-1} \circ \varphi_{j-i-1}^{-1} = \phi_j \circ \varphi_j^{-1} = \phi_i \circ (\phi_{j,i} \circ \varphi_{j,i}^{-1}) \circ \varphi_i.$$

By the requirement for  $(i, j)$ , we have  $\phi_{j,i} \circ \varphi_{j,i}^{-1} = \text{id}$ , so the anti-clockwise direction is  $\phi_i \circ \varphi_i$ , which is exactly the clockwise direction. So  $\beta$  is shown to be compatible with  $\phi_{j-i-1}$  and  $\varphi_{j-i-1}$ .  $\square$

We deduce some consequences from the above result.

**Theorem 5.6.** *Any isomorphism class of rank two semistable Higgs bundles with trivial chern classes over  $X$  is associated to an isomorphism class of crystalline representations of  $\pi_1(\mathbf{X}^0)$  into  $\text{GL}_2(k)$ . The image of the association contains all irreducible crystalline representations of  $\pi_1(\mathbf{X}^0)$  into  $\text{GL}_2(k)$ .*

*Proof.* The second statement follows from Theorem 5.2. Let  $(E, \theta)$  be a rank two semistable Higgs bundle with trivial  $c_1$  and  $c_2$  over  $X$ . By Theorems 2.6 and 2.5, it is a quasi-periodic Higgs bundle. Recall that we use the HN-filtration in the proof. Hence

we obtain *the* quasi-periodic Higgs-de Rham sequence for  $(E, \theta)$ . Let  $e \in \mathbb{N}_0$  be the minimal number such that  $(Gr_{HN} \circ C_0^{-1})^e(E, \theta)$  is periodic and say its period is  $f \in \mathbb{N}$ . Thus from  $(E, \theta)$  we obtain in the above way an object

$$((Gr_{HN} \circ C_0^{-1})^e(E, \theta), Fil_0 = HN, \dots, Fil_{f-1} = HN, \phi)$$

in  $\mathcal{HB}_{(0,f)}$ , which is unique up to the choice of  $\phi$ . Let  $\rho$  be the corresponding representation by Theorem 3.9. As  $HN$ s clearly satisfy the Assumption 5.3, it follows from Proposition 5.5 that the isomorphism class of  $\rho \otimes k$  is independent of the choice of  $\phi$ . It is clear that an isomorphic Higgs bundle to  $(E, \theta)$  is associated to the same isomorphism class of crystalline representations. This shows the first statement.  $\square$

Next, we want to compare the classical construction of Katz and Lange-Stuhler (see §4 [5] and §1 [7]) using an Artin-Schreier cover with the one in the current paper. Namely, we consider the isomorphism classes of vector bundles  $E$  over  $X$  satisfying  $F_X^{*f}E \cong E$  for an exponent  $f \in \mathbb{N}$ . By Proposition 1.2 and Satz 1.4 in [7] (see also §4.1 [5]), they are in bijection with the isomorphism classes of representations  $\pi_1(X) \rightarrow \mathrm{GL}(k)$ . Let  $[\rho_{KLS}]$  be the isomorphism class of representations  $\pi_1(X) \rightarrow \mathrm{GL}(k)$  corresponding to the isomorphism class of  $E$ . Let  $E$  be such a bundle over  $X$  with an isomorphism  $\phi : F_X^{*f}E \cong E$ . It gives rise to a tuple  $(E, 0, Fil_{tr}, \dots, Fil_{tr}, \phi)$ , an object in  $\mathcal{HB}_{(0,f)}$ . Then by Theorem 3.9, there is a corresponding crystalline representation  $\rho : \pi_1(\mathbf{X}^0) \rightarrow \mathrm{GL}(\mathbb{F}_{p^f})$ . After Proposition 5.5, the isomorphism class of  $\rho \otimes_{\mathbb{F}_{p^f}} k$  is independent of the choice of  $\phi$ . The following result follows directly from the construction of the representation due to Faltings [1].

**Lemma 5.7.** *Let  $\tau$  be a crystalline representation of  $\pi_1(\mathbf{X}^0)$  into  $\mathrm{GL}(\mathbb{F}_p)$  and  $(H, \nabla, Fil, \Phi)$  the corresponding object in  $\mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$ . If the filtration  $Fil$  is trivial, namely,  $Fil^0 H = H$ ,  $Fil^1 H = 0$ , then  $\tau$  factors through the specialization map  $sp : \pi_1(\mathbf{X}^0) \twoheadrightarrow \pi_1(X)$ .*

*Proof.* Let  $\mathbf{U}_i = \mathrm{Spec} R$  be a small affine subset of  $\mathbf{X}$ , and  $\Gamma = \mathrm{Gal}(\bar{R}|R)$  the Galois group of maximal extension of  $R$  étale in characteristic zero (cf. Ch. II. b) [1]). Let  $R^{ur} \subset \bar{R}$  be the maximal subextension which is étale over  $R$  and  $\Gamma^{ur} = \mathrm{Gal}(R^{ur}|R)$ . By the local nature of the functor  $\mathbf{D}$  (cf. Theorem 2.6 loc. cit.), it is to show that the representation  $\mathbf{D}(H_i)$  of  $\Gamma$ , constructed from the restriction  $H_i := (H, \nabla, Fil, \Phi)|_{\mathbf{U}_i} \in \mathcal{MF}_{[0,n]}^\nabla(R)$ , factors through the natural quotient  $\Gamma \twoheadrightarrow \Gamma^{ur}$ . To that we have to examine the construction of  $\mathbf{D}(H_i)$  carried in pages 36-39 loc. cit. (see also pages 40-41 for the dual object). First of all, we can choose a basis  $f$  of  $H_i$  which is  $\nabla$ -flat. Because  $Fil$  is trivial,  $\Phi$  is a local isomorphism. So for any basis  $e$  of  $H_i$ ,  $f = \Phi(e \otimes 1)$  is then a flat basis of  $H_i$ . The construction of module  $\mathbf{D}(H_i) \subset H_i \otimes \bar{R}/p$  does not use the connection, but the definition of  $\Gamma$ -action does (see page 37 loc. cit.). A basis of  $\mathbf{D}(H_i)$  is of form  $f \otimes x$ , where  $x$  is a set of tuples in  $\bar{R}/p$  satisfies the equation  $x^p = Ax$ , where  $A$  is the matrix of  $\Phi$  under the basis  $f$  (i.e.  $\Phi(f \otimes 1) = Af$ ). Now that  $A$  is invertible, the entries of  $x$  lie actually in  $R^{ur}/p$ . Since  $f$  is a flat basis, the action of  $\Gamma$  on  $f \otimes x$  coincides the natural action of  $\Gamma$  on the second factor. Thus it factors through the quotient  $\Gamma \twoheadrightarrow \Gamma^{ur}$ .  $\square$

By the above lemma,  $\rho$  factors as

$$\pi_1(\mathbf{X}^0) \xrightarrow{sp} \pi_1(X) \rightarrow \mathrm{GL}(\mathbb{F}_{p^f}).$$

**Theorem 5.8.** *Let  $\rho_F : \pi_1(X) \rightarrow \mathrm{GL}(\mathbb{F}_{p^f})$  be the induced representation from  $\rho$ . Then  $\rho_F \otimes k$  is in the isomorphism class  $[\rho_{KLS}]$ .*

*Proof.* We can assume that  $E$  as well as  $\phi$  are defined over  $X|k'$  for a finite field  $k'$ . Then we obtain from Proposition 4.1.1 [5] or Satz 1.4 [7] a representation  $\rho_{KLS} : \pi_1(X) \rightarrow \mathrm{GL}(\mathbb{F}_{p^f})$ . We are going to show that  $\rho_F$  and  $\rho_{KLS}$  are isomorphic  $\mathbb{F}_{p^f}$ -representations. For  $f = 1$ , this follows directly from their constructions: Katz and Lange-Stuhler construct the representation by solving  $\phi$ -invariant sections through the equation  $x^p = Ax$ , which it is exactly what Faltings does in the case of trivial filtration by the above description of his construction. For a general  $f$ , Katz and Lange-Stuhler solve locally the equation  $x^{p^f} = Ax$ , which is equivalent to a system of equations of form

$$x_0^p = x_1, \dots, x_{f-2}^p = x_{f-1}, x_{f-1}^p = Ax_0.$$

To examine our construction, we take a local basis  $e_0 = e$  of  $E_0 = E$  and put  $e_i = F_X^{*i}e$ , a local basis of  $E_i$  for  $0 \leq i \leq f-1$ . Write  $\phi(e_{f-1}) = Ae_0$ . Put  $\tilde{e} = (e_0, \dots, e_{f-1})$ , and  $\tilde{x} = (x_1, \dots, x_{f-1})$ . Then the  $\tilde{\phi}$  in Lemma 3.8 has the expression  $\tilde{\phi}(\tilde{e}) = \tilde{A}\tilde{e}$  with

$$\tilde{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ \phi & 0 & \cdots & 0 \end{pmatrix}.$$

One notices that the equation  $\tilde{x}^p = \tilde{A}\tilde{x}$  written into components is exactly the above system of equations. Thus one sees that the  $\mathbb{F}_{p^f}$ -representation  $\rho_F$  corresponding to  $(E, 0, \mathrm{Fil}_{tr}, \dots, \mathrm{Fil}_{tr}, \phi)$  by Corollary 3.9 is isomorphic to  $\rho_{KLS}$  as  $\mathbb{F}_{p^f}$ -representations.  $\square$

It may be noteworthy to deduce the following

**Corollary 5.9.** *Let  $\tau$  be a crystalline representation of  $\pi_1(\mathbf{X}^0)$  with the corresponding object  $(H, \nabla, \mathrm{Fil}, \Phi) \in \mathcal{MF}_{(0,n)}^\nabla(\mathbf{X}/W)$ . Then  $\tau$  factors through the specialization map iff the filtration  $\mathrm{Fil}$  is trivial.*

*Proof.* One direction is Lemma 5.7. It remains to show the converse direction. Let  $\tau_0$  be the induced representation of  $\pi_1(X)$  from  $\tau$ . As it is of finite image, one constructs directly from  $\rho_0$  a vector bundle  $E$  over  $X$  such that  $F_X^*E \cong E$ . Choosing such an isomorphism, we obtain a representation of  $\pi_1(X)$  and then a representation  $\tau'$  of  $\pi_1(\mathbf{X}^0)$  by composing with the specialization map. By Theorem 5.8,  $\tau' \otimes \mathbb{F}_{p^f}$  is isomorphic to  $\tau \otimes \mathbb{F}_{p^f}$  for a certain  $f \in \mathbb{N}$ . It follows from Proposition 3.10 (ii) that the filtration  $\mathrm{Fil}$  is trivial.  $\square$

We conclude the paper by providing many more examples beyond the rank two semistable Higgs bundles and strongly semistable vector bundles.

**Proposition 5.10.** *Let  $(H, \nabla, \mathrm{Fil}, \Phi) \in \mathcal{MF}_{[0,n]}^\nabla(\mathbf{X}/W)$ . Then any Higgs subbundle  $(G, \theta) \subset \mathrm{Gr}_{\mathrm{Fil}}(H, \nabla)$  of degree zero is strongly Higgs semistable with trivial chern classes.*

*Proof.* Put  $(E, \theta) = \mathrm{Gr}_{\mathrm{Fil}}(H, \nabla)$ . Proposition 0.2 [10] says that  $(E, \theta)$  is a semistable Higgs bundle of degree zero. Note that the operator  $\mathrm{Gr}_{\mathrm{Fil}} \circ C_0^{-1}$  does not change the degree, rank and definition field of  $(G, \theta)$ , and as there are only finitely many Higgs subbundles of  $(E, \theta)_0$  with the same degree, rank and definition field as  $(G, \theta)$ , there exists a pair  $(e, f)$  of nonnegative integers with  $s > r$  such that

$$(\mathrm{Gr}_{\mathrm{Fil}} \circ C_0^{-1})^s(G, \theta) = (\mathrm{Gr}_{\mathrm{Fil}} \circ C_0^{-1})^r(G, \theta)$$

holds. Thus  $(G, \theta)$  is quasi-periodic and strongly Higgs semistable with trivial chern classes by Theorem 2.5.  $\square$

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